
PRICING OF MOVING- AVERAGE-TYPE OPTIONS WITH APPLICATIONS

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Moving-average-type options are complex path-dependent derivatives whose payoff depends on the moving average of stock prices. This article concentrates on two such options traded in practice: the moving-average-lookback option and the moving-average-reset option. Both options were issued in Taiwan in 1999, for example. The moving-average-lookback option is an option struck at the minimum moving average of the underlying asset's prices. This article presents efficient algorithms for pricing geometric and arithmetic moving-average-lookback options. Monte Carlo simulation confirmed that our algorithms converge quickly to the option value. The price difference between geometric averaging and arithmetic averaging is small. Because it takes much less time to price the geometric-moving-average version, it serves as a practical approximation to the arithmetic moving-average version. When applied to the moving-average-lookback options traded on Taiwan's stock exchange, our algorithm gave

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almost the exact issue prices. The numerical delta and gamma of the options revealed subtle behavior and had implications for hedging. The moving-average-reset option was struck at a series of decreasing contract-specified prices on the basis of moving averages. Similar results were obtained for such options with the same methodology. © 2003 Wiley Periodicals, Inc. *Jrl Fut Mark* 23:415–440, 2003

INTRODUCTION

Path-dependent derivatives are derivative securities whose payoff depends nontrivially on the price history of the underlying asset. Some path-dependent derivatives such as lookback and reset options can be efficiently priced. Others, however, are difficult to price as surveyed in Lyuu (2002). Options based on the arithmetic average of the underlying asset's prices—such as the (arithmetic) Asian option—are prominent examples. As these options typically lack simple closed-form solutions, the development of numerical algorithms is essential.

This article focuses on moving-average-type options. They are path-dependent derivatives whose payoff depends on the moving average of stock prices. For many investors, moving average is a popular technical measure for a short-term trend in, or even a fair value of, the stock price. This article concentrates on two such options found in practice: the moving-average-lookback option and the moving-average-reset option. The moving-average-lookback option is struck at the minimum moving average of the underlying stock prices over a contractual period. The moving-average-reset option is similar. It is struck at a series of decreasing contract-specified prices over a contractual period on the basis of the moving averages. These two options were traded on Taiwan's exchange, for example. In practice, both options use arithmetic average and are American style. Our algorithmic methodology is the same for both options and is applicable to a wide variety of moving-average-type options.

The reset feature makes the option useful in portfolio insurance because it protects the investors amidst stock price declines. The advantages of using a moving average instead of the stock price alone, as in ordinary reset options, are to mitigate the possibility of stock price manipulation, especially for thinly traded markets; provide a strike price correlated with a perceived price trend or fair value; lower the option price as compared with ordinary reset options; and mitigate the delta jump inherent in certain reset options. These advantages make the products appealing to some investors. The price that comes with them is complexity.

TABLE I
Listed Options Market in Taiwan: 1997–2001

Year	Issue Premium (Millions TWD)	Issue Volume (Millions of Shares)	Trading Volume per Year (Millions TWD) (Millions of Shares)	
1997	4,032	204	1,960	165
1998	4,743	270	13,069	1,492
1999	13,381	866	64,782	3,807
2000	12,668	1,110	162,262	11,588
2001	4,772	893	28,440	7,784

Note. These options were called warrants because they were issued in fixed quantities. They were listed by the Taiwan Stock Exchange Corporation (TSEC). Options in Taiwan are dividend-protected. TWD: New Taiwan dollar.

Taiwan's listed options market was born in 1997. Table I documents its trading activities over the 1997–2001 period. Options whose strike prices are related to the arithmetic moving average of the underlying stock prices were first issued in 1997. The issuance of options with similar features was particularly active in 1999 and involved all the major players in the capital market. Among them, the most prominent examples are the moving-average-lookback options and the moving-average-reset options, the focus of this article. Moving-average-type options made up a significant portion of the listed options market in 1999. In that year, a total of 13 moving-average-type options were issued. Their combined premium stood at 3.07 billion TWD, which was nearly 23% of the market (see Table II).¹

Pricing moving-average-type options is an intricate problem. The moving-average-lookback option contains the features of Asian, lookback, and reset options, whereas the moving-average-reset option contains the features of Asian and reset options. It is the combination of features and the adoption of a moving average that make the options so difficult to price. This article concentrates on the moving-average-lookback option (MAL). The slightly simpler moving-average-reset option (MAR) can be handled by the same methodology and will receive less coverage. At any rate, the general conclusions for MARS are similar to those for MALs.

Throughout this article, S_t denotes the stock price at date t . Let m_t be the six-day moving average at date t , where $t \geq 5$; thus

$$m_t = \frac{S_{t-5} + S_{t-4} + S_{t-3} + S_{t-2} + S_{t-1} + S_t}{6}$$

¹The average exchange rate in 1999 was 32.27 TWD/USD.

TABLE II
Moving-Average-Type Options Issued in 1999 on the Taipei Stock Exchange

<i>Contract</i>	<i>Type</i>	<i>Issue Date</i>	<i>Issue Volume</i> (Thousands of Shares)	<i>Issue Premium</i> (Millions TWD)
GC06	Reset	28-Apr	10,700	216.68
GC07	Reset	27-May	17,500	239.40
GC08	Reset	09-Jun	10,000	248.30
GC09	Reset	14-Jun	13,600	206.86
GC10	Reset	20-Oct	12,000	262.44
YT07	Reset	23-Nov	22,000	226.64
NS02	Reset	16-Jun	10,000	200.00
NS03	Reset	13-Sept	10,000	205.00
FB01	Reset	08-Jul	20,000	220.00
FB02	Reset	18-Aug	18,000	306.00
CS04	Reset	04-Sept	11,700	216.74
PL06	Lookback	21-Aug	10,000	269.80
PL07	Lookback	27-Aug	15,000	251.40

Note. All the options have a maturity of one year. To put the numbers in perspective, the total dollar amount of options issued in 1999 was 13.4 million TWD. GC: Grand Cathay Securities (capital 10.5 billion TWD), YT: Yuanta Securities (capital 11.7 billion TWD), NS: National Securities (capital 8 billion TWD), FB: Fubon Securities (capital 10.5 billion TWD), CS: Capital Securities (capital 9.2 billion TWD), and PL: Polaris Securities (capital 6 billion TWD). The capitalization figures are based on 1999 filings.

Given the previous day's moving average

$$m_{t-1} = \frac{S_{t-6} + S_{t-5} + S_{t-4} + S_{t-3} + S_{t-2} + S_{t-1}}{6}$$

m_t remains dependent on the earlier price S_{t-6} because

$$m_t = m_{t-1} - \frac{S_{t-6} - S_t}{6}$$

This shows that process (m_t, S_t) is not Markovian. In contrast, the averaging process (A_t, S_t) most relevant in pricing Asian options, where

$$A_t \equiv \frac{\sum_{i=0}^t S_i}{t+1}$$

is Markovian because

$$A_t = \frac{tA_{t-1} + S_t}{t+1}$$

Recall that the payoff of the Asian option is a function of A_m at expiration date m . The moving average therefore introduces complexity not present in Asian options.

Although process (m_t, S_t) is not Markovian, the six-day moving window $(S_{t-5}, S_{t-4}, S_{t-3}, S_{t-2}, S_{t-1}, S_t)$ clearly is. It is a moving window

instead of the fixed window used by A_t because the starting date of the six-day averaging window moves with the passage of time. The six-day stock prices certainly suffice to calculate the six-day moving average m_t . The main reason the moving-averaging process (m_t, S_t) is non-Markovian is, therefore, that the *order* of the six stock prices comprising m_t counts. However, keeping track of the order of individual stock prices greatly enlarges the state space. This article develops practical resolution-reduction methods to lower the amount of information needed to encode the states.

Absent analytic formulae, the Monte Carlo method is always available [see Boyle, Broadie, & Glasserman (1997)]. However, it suffers from the inability to handle early exercise. Recently, Longstaff and Schwartz (2001) have developed a least-squares Monte Carlo approach to tackle the early-exercise problem. A common disadvantage of both methods is that the resulting option value is probabilistic. The alternative deterministic tree method can handle early exercise. Its challenge lies in making it efficient. This article develops tree-based algorithms on the basis of the CRR binomial model (Cox, Ross, & Rubinstein, 1979). Monte Carlo simulation is used to verify the calculated prices. Besides being accurate, the algorithms converge quickly to the option value. Interestingly, the price difference between geometric and arithmetic MALs is very small. Because it takes much less time to price the geometric version, it is a practical approximation to the arithmetic version.

When the algorithms are applied to the actual contracts traded in Taiwan, their issue prices are essentially obtained exactly. Such excellent matches in market prices independently confirm the accuracy and practicality of the methodology. Subsequent data after the issue date exhibit patterns of volatility qualitatively typical of Taiwan's traded securities. Sensitivity measures are needed for hedging purposes. Numerical data reveal subtle behavior in deltas and gammas for moving-average-type options. They have implications for traders engaged in risk management and hedging.

There is scarcely any prior work on the pricing of moving-average-type options. Cheng and Zhang (2000) discuss an option similar to MAL. However, it is based on geometric, continuous averaging over a fixed window instead of arithmetic, discrete averaging over a moving window found in practice and treated in this article. Zvan, Vetzal, and Forsyth (1999) discuss the discrete Asian barrier option, but again it is based on a fixed window. Babbs (2000) and Cheuk and Vorst (1997) discuss lookback options, Gray and Whaley (1997) analyze the S&P 500 bear market warrant with a single reset, and Hull and White (1993) and Klassen (2001) propose

several approximation algorithms to price Asian options. These works do not translate directly into solutions to moving-average-type options because they address only one aspect of the complexities; furthermore, none of them address moving averages. Chang, Chung, and Shackleton (2000) use the Hull–White paradigm to price a MAR-like option. However, they use fixed windows, and the state-reducing interpolation scheme lacks convergence guarantees as mentioned in Forsyth, Vetzal, and Zvan (2001).

The remainder of this article is organized as follows. The models are succinctly reviewed in a quick review of the models. The third section covers the pricing of geometric MALS. The fourth section covers the pricing of arithmetic MALS. The fifth section discusses the numerical delta and gamma of MALS. The sixth section conducts empirical studies of two arithmetic MALS and two arithmetic MARS traded on the Taipei Stock Exchange. The final section concludes this article. Programs are available at www.csie.ntu.edu.tw/~lyuu-jfm.zip.

QUICK REVIEW OF THE MODELS

The stock price is assumed to follow the lognormal diffusion

$$dS/S = (r - q)dt + \sigma dW$$

in a risk-neutral economy with a dividend yield of q . The CRR binomial model will be adopted to approximate the stock's price dynamics. This binomial model converges to the continuous-time model, which is well known [see, e.g., Heston & Zhou (2000)].

The binomial model has parameters u and d with $u > d$. From any node with stock price S , price Su follows with probability p (the up move), and price Sd follows with probability $1 - p$ (the down move). If the binomial model has n periods, covering T years, then $u = e^{\sigma\sqrt{T/n}}$, $d = 1/u$, and $p = [e^{(r-q)T/n} - d]/(u - d)$.

Let node (i, j) , $0 \leq j \leq i \leq n$, stand for the node at time i with j cumulative up moves and, by implication, $i - j$ cumulative down moves. Node (i, j) is reached with probability $\binom{i}{j} p^j (1 - p)^{i-j}$. The resulting topology has $\sim n^2/2$ nodes and is called the binomial tree.

Although MAL and MAR have the same payoff function as the vanilla call, their strike prices may be lowered in the future. Therefore, they cannot cost less than an otherwise identical vanilla call. An American-style vanilla call costs more than its exercise payoff before the expiration date if the underlying stock pays no dividends or is dividend-protected. Because the American-style MAL and MAR cost as much as an otherwise

identical vanilla call, they also cost more than the exercise payoff. They will not be exercised early under the same conditions. We conclude that both the American-style MAL call and the American-style MAR call can be priced as if they were European style if the underlying stock pays no dividends or is dividend-protected.

This article focuses on calls because puts can be treated similarly. Unless stated otherwise, all options are calls.

PRICING GEOMETRIC MAL

This section examines the geometric moving-average-lookback option (GMAL). The purpose is twofold. First, a methodology is developed to price the geometric averaging version. Second, after the methodology's accuracy is verified, it is applied to the arithmetic case. The arithmetic and the geometric cases produce very close prices.

Defining GMAL

Daily closing prices are used in calculating moving averages. Let $0, 1, \dots, n$ be the $n + 1$ trading dates on or before the reset date T_s (in years), that is, date n . When calculating an a -day moving average, a stock prices are involved. The a -day moving product at date i is defined as the product of the a prices up to date i

$$S_{i-a+1}S_{i-a+2}\dots S_i = \prod_{j=i-a+1}^i S_j, \quad a - 1 \leq i \leq n$$

The a -day geometric moving average at date i is then $(\prod_{j=i-a+1}^i S_j)^{1/a}$. The minimum a -day geometric moving average as of the reset date is therefore

$$m_a \equiv \min \left[\left(\prod_{i=0}^{a-1} S_i \right)^{1/a}, \left(\prod_{i=1}^a S_i \right)^{1/a}, \dots, \left(\prod_{i=n-a+1}^n S_i \right)^{1/a} \right]$$

The payoff of the European-style GMAL (EGMAL) at expiration date $m > n$ is $\max(S_m - X, 0)$ with the strike price

$$X = \max[\min(m_a, \text{UB}), \text{LB}]$$

The seemingly complex strike price is easy to explain. It is struck at the minimum a -day moving average but must be banded between LB and UB. LB sets the lower bound on the strike price, and UB the upper bound. Both LB and UB are written in the contract. In Taiwan, for example, UB is

set to S_0 , and LB is some proportion (e.g., 90%) of UB . The American-style GMAL is identical except that in determining the strike price X used at date i , the moving average m_a should be the minimum a -day geometric moving average up to date i .

Setup of the Binomial Tree

A key feature of the a -day geometric average on the binomial tree is that it must be $S_0 u^{k/a}$ for some integer k . Let k_{UB} be the smallest integer k such that $S_0 u^{k/a} \geq UB$ and k_{LB} be the largest integer k such that $S_0 u^{k/a} \leq LB$. The set of potential moving averages for strike prices is then

$$\{S_0 u^{k/a} : k \text{ is an integer, } k_{LB} \leq k \leq k_{UB}\}$$

which contains $k_{UB} - k_{LB} + 1$ numbers.

Suppose the binomial tree partitions each day into L periods. The number of periods before the reset date is therefore nL . GMAL is identical to the vanilla call at expiration with the strike price determined earlier at the reset date. Therefore, to price EGMALS, the tree is built up to the reset date, and then the Black–Scholes formula is plugged in at the terminal nodes. The tree contains $\sim (nL)^2/2 = O(n^2L^2)$ nodes.

The number of nodes can be reduced as follows. Observe that moving averages involve only daily closing prices. Although the tree covers nL periods, only nodes falling on dates $0, 1, 2, \dots, n$ are essential. In fact, the (nL) -period tree can be made n period without losing information, with one period per day instead of L . The idea is to keep only those nodes on the dates. Each node is then linked to those nodes that can be reached in L periods on the original binomial tree. A node now has $L + 1$ successor nodes. This creates an n -period $(L + 1)$ -ary tree with $O(n^2L)$ nodes instead of the preceding $O(n^2L^2)$. For example, when $L = 2$ a trinomial tree with half as many periods is created, as shown in Figure 1. Every branch on the trinomial tree corresponds to two branches on the binomial tree. As another example, when $L = 6$ a heptanomial tree with one-sixth as many periods results, as depicted in Figure 2.

Let $N(i, j)$ denote the node on the $(L + 1)$ -ary tree at dates i with j cumulative up moves (hence, $i - j$ down moves by implication) on the binomial tree. Let $S(i, j)$ denote the stock's price at node $N(i, j)$. Node $N(i, j)$ has $L + 1$ successor nodes

$$N(i + 1, j), N(i + 1, j + 1), \dots, N(i + 1, j + L)$$

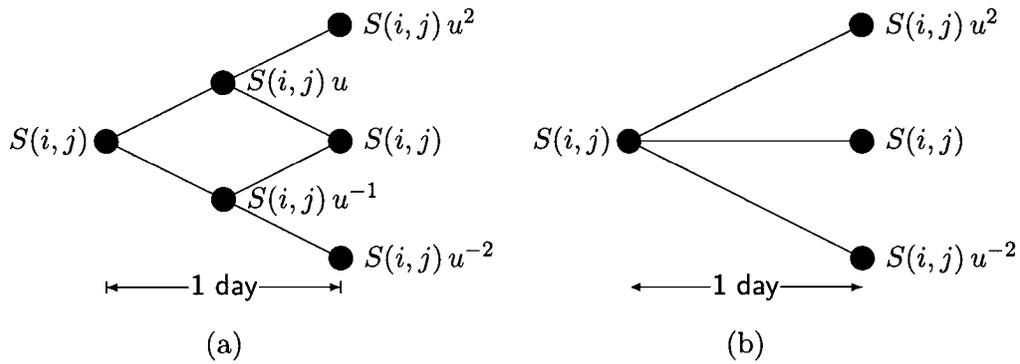


FIGURE 1

Turning a binomial tree into an $(L + 1)$ -ary tree, $L = 2$: (a) a two-period binomial tree covering one day and (b) a one-period trinomial tree covering one day after the intermediate nodes are removed from the binomial tree of (a). A period for the trinomial tree lasts twice as long as that for the binomial tree. A down move on the trinomial tree (b) corresponds to two down moves on the binomial tree (a). A flat move on the trinomial tree corresponds to a total of one up move and one down move on the binomial tree. An up move on the trinomial tree corresponds to two up moves on the binomial tree.

In particular, the ℓ th branch takes node $N(i, j)$ to node $N(i + 1, j + \ell)$. The probability that this branch is followed is

$$p(\ell) \equiv \binom{L}{\ell} p^\ell (1 - p)^{L - \ell}$$

because it corresponds to ℓ up moves and $L - \ell$ down moves on the original binomial tree. Their stock prices are related by

$$S(i + 1, j + \ell) = S(i, j)u^\ell d^{L - \ell} = S(i, j)u^{2\ell - L} \tag{1}$$

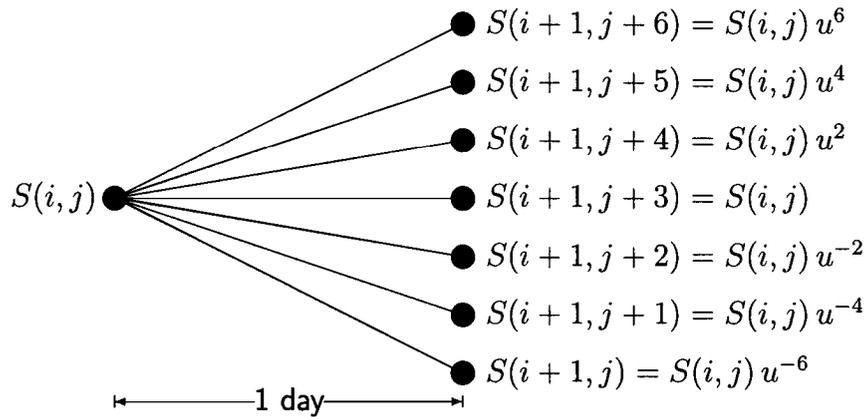


FIGURE 2

Turning a binomial tree into an $(L + 1)$ -ary tree, $L = 6$. Because a six-period binomial tree has seven terminal prices, a heptanomial tree is obtained. For example, $S(i + 1, j + 4)$ is reached by taking four up moves (hence two down moves) on the original binomial tree.

Nodes on the $(L + 1)$ -ary tree must be augmented with states besides the stock price. Each state at date t records the a daily stock prices involved in the average at date t , that is

$$(S_{t-a+1}, S_{t-a+2}, \dots, S_t)$$

When the next stock price S_{t+1} appears, the new state becomes

$$(S_{t-a+2}, S_{t-a+3}, \dots, S_{t+1})$$

a simple left-shifting operation. Because S_{t+1} is the result of taking one of the $L + 1$ branches from S_t , a number between 0 and L suffice to represent that transition. Any state $(S_{t-a+1}, S_{t-a+2}, \dots, S_t)$ contains $a - 1$ transitions; therefore, it can be represented by a tuple $\overline{s_1 s_2 \dots s_{a-1}}$, where $s_i \in \{0, 1, \dots, L\}$. The number s_i signifies that stock price S_{t-a+i} takes the s_i th branch to reach price $S_{t-a+i+1}$. By Equation (1)

$$S_{t-a+i+1} = S_{t-a+i} u^{2s_i-L} \tag{2}$$

Take $L = 2$ and $a = 3$, for example. The tree is trinomial. Suppose 0, 1, and 2 mean down, flat, and up movements, respectively (recall Fig. 1). The state $\overline{12}$ means a flat move and then an up move have been taken to reach the current state. Table III tabulates the $(2 + 1)^2 = 9$ states and their interpretations.

Rigorously, state $(S_{t-a+1}, S_{t-a+2}, \dots, S_t)$ should be encoded by

$$(\overline{s_1 s_2 \dots s_{a-1}}, S_t)$$

TABLE III
Encoding Two-Day Movements

State	Moves on the Binomial Tree	
$\overline{00}$	0 Up move	4 Down moves
$\overline{01}$	1 Up move	3 Down moves
$\overline{02}$	2 Up move	2 Down moves
$\overline{10}$	1 Up moves	3 Down moves
$\overline{11}$	2 Up move	2 Down moves
$\overline{12}$	3 Up moves	1 Down moves
$\overline{20}$	2 Up move	2 Down moves
$\overline{21}$	3 Up moves	1 Down move
$\overline{22}$	4 Up moves	0 Down move

Note. Graphically from Figure 1(b), 0, 1, and 2 mean down, flat, and up movements, respectively. Figure 1(a) gives a more complete picture: 0 means 2 down moves, 1 means a total of 1 down move and 1 up move, and 2 means 2 up moves, all referring to the original binomial tree.



instead of merely $\overline{s_1 s_2 \cdots s_{a-1}}$. S_t is often dropped because stock price S_t is often implicit and because ambiguity is seldom an issue. Given state $(\overline{s_1 s_2 \cdots s_{a-1}}, S_t)$, the $a - 1$ prices preceding S_t are, in reverse chronological order

$$S_t u^{L-2s_{a-1}}, S_t u^{2L-2(s_{a-1}+s_{a-2})}, \dots, S_t u^{(a-1)L-2(s_{a-1}+s_{a-2}+\cdots+s_1)}$$

according to Relation (2). The a -day geometric average for the state is therefore

$$\begin{aligned} & S_t [u^{L-2s_{a-1}} u^{2L-2(s_{a-1}+s_{a-2})} \dots u^{(a-1)L-2(s_{a-1}+s_{a-2}+\cdots+s_1)}]^{1/a} \\ & = S_t u^{(a-1)L/2-2[s_{a-1}(a-1)+s_{a-2}(a-2)+\cdots+s_1]/a} \end{aligned}$$

Recall that state transition is accomplished by left-shifting of the state

$$\overline{s_1 s_2 \cdots s_{a-1}} \rightarrow \overline{s_2 s_3 \cdots s_a}$$

The leftmost number s_1 is dropped as the new number s_a is shifted in from the right. If b is the current state and the transition takes the ℓ th branch, then the next state is denoted by $\text{NEXT}(b, \ell)$. See Table IV for the complete tabulation for the case of a three-day moving average with $L = 2$.

By making the stock price component of a state explicit, state transition

$$(\overline{s_1 s_2 \cdots s_{a-1}}, S_t) \rightarrow (\overline{s_2 s_3 \cdots s_a}, S_{t+1}) \tag{3}$$

TABLE IV
State Transition for Two-Day Movements

Current State b	One-Day Forward $\text{NEXT}(b, \ell)$
$\overline{00}$	$\overline{0\ell}$
$\overline{01}$	$\overline{1\ell}$
$\overline{02}$	$\overline{2\ell}$
$\overline{10}$	$\overline{0\ell}$
$\overline{11}$	$\overline{1\ell}$
$\overline{12}$	$\overline{2\ell}$
$\overline{20}$	$\overline{0\ell}$
$\overline{21}$	$\overline{1\ell}$
$\overline{22}$	$\overline{2\ell}$

Note. The branch taken is ℓ . The new state results from dropping the leftmost number and appending ℓ from the right. The table assumes $a = 3$ and $L = 2$.



can be used to calculate the new a -day moving average. Here, S_{t+1} is the result of S_t taking the s_a th branch. Suppose the moving average for the left state is A . Then the new moving average is

$$\begin{aligned} \left[\frac{A^a S_{t+1}}{S_t u^{(a-1)L-2(s_{a-1}+s_{a-2}+\dots+s_1)}} \right]^{1/a} &= \left[\frac{A^a}{u^{aL-2(s_a+s_{a-1}+s_{a-2}+\dots+s_1)}} \right]^{1/a} \\ &= \frac{A}{u^{L-2(s_a+s_{a-1}+s_{a-2}+\dots+s_1)/a}} \end{aligned} \quad (4)$$

by Relation (2). Define function

$$\text{MA}[\overline{s_1 s_2 \dots s_{a-1}}, S_t, s_a] = k$$

to denote state Transition (3) when the new average is $S_0 u^{k/a}$. The computation of $\text{MA}(\)$ can be based on Formula (4).

Take $L = 2$ and $a = 3$, for example. Consider node $N(i, j)$ with stock price $S_0 u^{2j-iL}$ and state $\overline{12}$ from Table III. The state is the result of a flat move followed by an up move. After backtracking those moves, node $N(i, j)$ is reached from node $N(i-1, j-2)$ at date $i-1$ and node $N(i-2, j-3)$ at date $i-2$. The associated prices are $S_0 u^{2(j-2)-(i-1)L}$ and $S_0 u^{2(j-3)-(i-2)L}$, respectively. The three-day moving average for the state is therefore

$$\{S_0^3 u^{[6j-3L(i-1)-10]}\}^{1/3} = S_0 u^{[6j-3L(i-1)-10]/3}$$

Pricing European-Style GMAL

Let $C(i, j; b, k)$ denote the option price on node $N(i, j)$ with path-dependent state (b, k) . The parameter b encodes the state. However, b alone is not sufficient for pricing purposes; the prevailing strike price is also needed. The integer k signifies that the minimum a -day moving product of the stock prices up to date i is $S_0^a u^k$ (equivalently, the moving average is $S_0 u^{k/a}$). By contract, this number is bounded: $k_{LB} \leq k \leq k_{UB}$. This feature helps reduce the state space. Forward induction can determine the minimum and maximum k precisely, denoted by $\min k_{i,j}$ and $\max k_{i,j}$, respectively, for each node $N(i, j)$.

The minimum geometric moving average may change after each transition. Consider the transition from node $N(i, j)$ with state b and minimum moving average $S_0 u^{k/a}$ to node $N(i+1, j+\ell)$, by taking the ℓ th branch. Let $x(b, \ell, k)$ be such that $S_0 u^{x(b, \ell, k)/a}$ is the new minimum a -day moving average. By considering how the minimum

moving average is affected if the new moving average is not smaller than the current minimum moving average, the new moving average is between lower bound LB and the current minimum moving average, and the new moving average is smaller than LB , the following formula is obtained:

$$x(b, \ell, k) \equiv \begin{cases} k, & \text{if } k \leq MA(b, \ell) \\ MA(b, \ell), & \text{if } k_{LB} \leq MA(b, \ell) < k \\ k_{LB}, & \text{if } MA(b, \ell) < k_{LB} \end{cases}$$

The function $x(b, \ell, k)$ that updates the prevailing strike price is straightforward to calculate.

Ground has been laid for the pricing algorithm. The terminal payoffs are given by the Black–Scholes formula for all combinations of stock price $S(n, j)$ and strike price $S_0 u^{k/a}$, where $\min k_{n,j} \leq k \leq \max k_{n,j}$. Inductively, the backward-induction formula is

$$C(i, j; b, k) = \frac{\sum_{\ell=0}^L p(\ell) C[i+1, j+\ell; \text{NEXT}(b, \ell), x(b, \ell, k)]}{R} \quad (5)$$

where $\min k_{i,j} \leq k \leq \max k_{i,j}$ and R represent the gross riskless return per day. The option value appears in $C(0, 0; 0, 0)$.

The algorithm for the EGMAL runs in $O(n^2 L^{a+3})$ time. Here is the analysis. There is a total of $O(n^2 L)$ nodes. Each node contains $(L+1)^{a-1}$ possible b 's and $O(L^2)$ possible k 's. Formula (5) takes $O(L)$ time to compute. These four bounds are multiplied to obtain the time bound. As the time bound is exponential in a , both a and L cannot be large. This is typically the case in practice.

Pricing American-Style GMAL

For American-style GMALS, two changes must be made. First, the option holder can exercise the option on or before the reset date. Early exercise is addressed by taking the maximum of the continuation value $C(i, j; b, k)$ and the exercise payoff $S_0 u^{2j-iL} - S_0 u^{k/a}$. Second, after the reset date when the strike price is set, the option is simply a vanilla American-style call. The Black–Scholes formula from each state on a terminal node should be replaced with the binomial tree algorithm for American-style options.

TABLE V
Pricing EGMALS

LB	σ	$a = 3, L = 8$			$a = 5, L = 3$		
		CRR	MC	SE	CRR	MC	SE
45	0.3	6.1689	6.1712	(0.0019)	6.0769	6.0745	(0.0019)
	0.4	8.1916	8.1942	(0.0029)	8.0924	8.0871	(0.0028)
	0.5	10.1367	10.1392	(0.0039)	10.0360	10.0339	(0.0038)
40	0.3	6.2694	6.2723	(0.0018)	6.1566	6.1521	(0.0018)
	0.4	8.4219	8.4242	(0.0026)	8.2832	8.2797	(0.0027)
	0.5	10.4953	10.4992	(0.0036)	10.3402	10.3332	(0.0036)
35	0.3	6.2714	6.2731	(0.0018)	6.1579	6.1551	(0.0018)
	0.4	8.4414	8.4460	(0.0026)	8.2970	8.2922	(0.0026)
	0.5	10.5581	10.5604	(0.0035)	10.3882	10.3836	(0.0035)

Note. The parameters are $S_0 = UB = 50$, $r = 2\%$, $q = 4\%$, $T = 1$, and $T_s = 1/12$ ($n = 22$). SE is the standard error of Monte Carlo simulation (MC) on the basis of 1,000,000 paths: 500,000 plus 500,000 antithetic.

Numerical Results

The binomial tree and Monte Carlo simulation are used to price the three-day and five-day EGMALS. Assume $S_0 = UB = 50$, $r = 2\%$, $q = 4\%$, $T = 1$ (year), and $T_s = 1/12$ (year). Suppose there are 22 trading days in a month; therefore, $n = 22$. LB, σ , and the moving-average length a will be varied. Fix $L = 8$ for the three-day cases ($a = 3$) and $L = 3$ for the five-day cases ($a = 5$). The pricing results by Monte Carlo simulation are based on 1,000,000 paths: 500,000 plus 500,000 antithetic.

The numerical results are tabulated in Table V. Two observations are made. First, the prices calculated by the algorithm are within two times the standard error generated by Monte Carlo simulation. The accuracy of the algorithm is therefore ensured. Second, empirically, the option value decreases with the moving-average length a and LB, but it increases with σ .

The convergence behavior is illustrated in Figure 3. The price converges quickly, with accuracy up to two decimal places when the number of periods per day, L , is at least 3. The pattern of convergence oscillates in a regular manner.

PRICING ARITHMETIC MAL

This section applies the same methodology to price the arithmetic moving-average-lookback option (AMAL). AMAL is similar to GMAL except

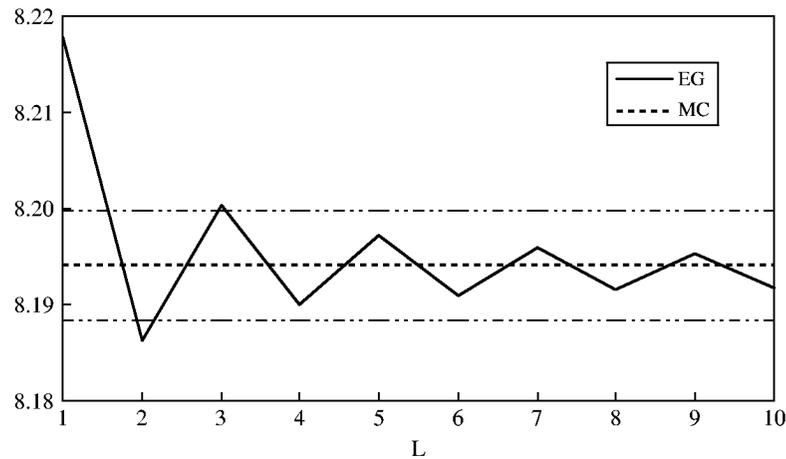


FIGURE 3

Convergence of the EG algorithm. The parameters are $S_0 = UB = 50$, $\sigma = 40\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $n = 22$, and $a = 3$. MC represents the prices obtained by Monte Carlo simulation on the basis of 1,000,000 paths: 500,000 plus 500,000 antithetic. Here, MC gives a value of 8.1942 with a standard error of 0.0029. A band with a width of two standard errors above and below MC is plotted for reference.

that the geometric moving average is replaced with the arithmetic average

$$m_a \equiv \min_{a-1 \leq t \leq n} \frac{\sum_{i=t-a+1}^t S_i}{a}$$

Because of the nonlognormality of the arithmetic sum of stock prices, the same complexity issue facing Asian options reappears.

Pricing AMAL on the Binomial Tree

A strike price must be some moving average. In preparation for the algorithm, all the possible strike prices between LB and UB are enumerated and put in table M , sorted from the smallest to the largest

$$LB \leq M(1) \leq \dots \leq M(k_{\max}) \leq UB$$

where k_{\max} is the size of M . The possible strike prices in the arithmetic version therefore form the set

$$\{M(k) : k \text{ is an integer, } 1 \leq k \leq k_{\max}\}$$

The basic computational structure remains the same as in the geometric version. Let $C(i, j; b, k)$ denote the option price on node $N(i, j)$ with path-dependent state (b, k) . The number b is the same as in the GMAL case. Integer k is an index into the table M such that $M(k)$ is the prevailing strike price. As M grows exponentially in a , its size should be reduced further. The algorithm simply rounds the strike prices in M to 3 decimal places.² The reduced resolution has the effect of equating strike prices in M that are close to each other such as 20.0001 and 20.0004. Take $LB = 45$ and $UB = 50$, for example. The set of possible strike prices is a subset of $\{45.000, 45.001, \dots, 50.000\}$, a total of 5001 numbers. The size of M can thus be treated as a constant.

Let $MA(b, \ell)$ denote the new moving average after taking the ℓ th branch at node $N(i, j)$ with state b . For any moving average y , let function $f(y)$ give the index k of (rounded) y in table M , that is, $M[f(y)] = y$. Such a function can be easily computed and put in a table for fast lookup. The backward-induction formula for AMAL is identical to that for GMAL in Formula (5) except for the definition of $x(b, \ell, k)$. This function now becomes

$$x(b, \ell, k) = \begin{cases} k, & \text{if } M(k) \leq MA(b, \ell) \\ f[MA(b, \ell)], & \text{if } LB \leq MA(b, \ell) < M(k) \\ 1, & \text{if } MA(b, \ell) < LB \end{cases}$$

The explanation is the same as before. The overall pricing structure thus mirrors that in the geometric case.

The algorithm runs in $O(n^2L^{a+1})$ time. Although the time bound seems lower than that for GMAL, the opposite is typically true in practice. The reason is the big constant factor in the running time.

Numerical Results

With the same parameters as in the geometric version, Table VI demonstrates that the prices of European-style AMALS (EAMALS) are close to the preceding geometric counterparts in Table V. The convergence is quite fast as shown in Figure 4. The price of GMAL is greater than that of an otherwise identical AMAL because of the smaller mean of the geometric average. This claim is proved rigorously in Fang (2002). Although their

²The idea comes from the fact that quoted prices are rounded to two decimal places in the markets. It turns out that three decimal places already demonstrate excellent accuracy although some information is lost. See Dai and Lyuu (2002) for a more elaborate approach.

TABLE VI
Pricing EAMALS

LB	σ	$a = 3, L = 8$			$a = 5, L = 3$		
		CRR	MC	SE	CRR	MC	SE
45	0.3	6.1684	6.1706	(0.0019)	6.0757	6.0726	(0.0019)
	0.4	8.1909	8.1933	(0.0029)	8.0907	8.0864	(0.0028)
	0.5	10.1358	10.1380	(0.0039)	10.0340	10.0314	(0.0038)
40	0.3	6.2688	6.2715	(0.0018)	6.1552	6.1507	(0.0018)
	0.4	8.4209	8.4225	(0.0026)	8.2809	8.2775	(0.0027)
	0.5	10.4937	10.4987	(0.0036)	10.3371	10.3299	(0.0036)
35	0.3	6.2708	6.2724	(0.0018)	6.1564	6.1542	(0.0018)
	0.4	8.4404	8.4449	(0.0026)	8.2946	8.2881	(0.0026)
	0.5	10.5563	10.5573	(0.0035)	10.3847	10.3812	(0.0035)

Note. The parameters are $S_0 = U_B = 50$, $r = 2\%$, $q = 4\%$, $T = 1$, and $T_s = 1/12$ ($n = 22$). SE is the standard error of Monte Carlo simulation (MC) on the basis of 1,000,000 paths: 500,000 plus 500,000 antithetic.

price difference increases with the moving-average period a and volatility σ , it remains small. Figure 5 demonstrates that the difference is quite stable when $L \geq 3$. As it takes much less time to price GMAL, it is a good approximation to AMAL in practice.

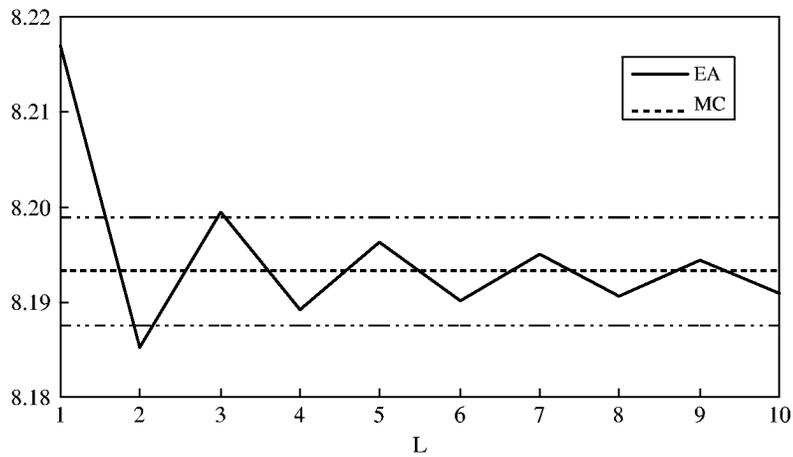


FIGURE 4

Convergence of the EAMAL algorithm. The parameters are $S_0 = U_B = 50$, $\sigma = 40\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), and $a = 3$. MC represents the prices obtained by Monte Carlo simulation on the basis of 1,000,000 paths: 500,000 plus 500,000 antithetic. It gives a value of 8.1933 with a standard error of 0.0029. A band with a width of two standard errors above and below MC is plotted for reference.

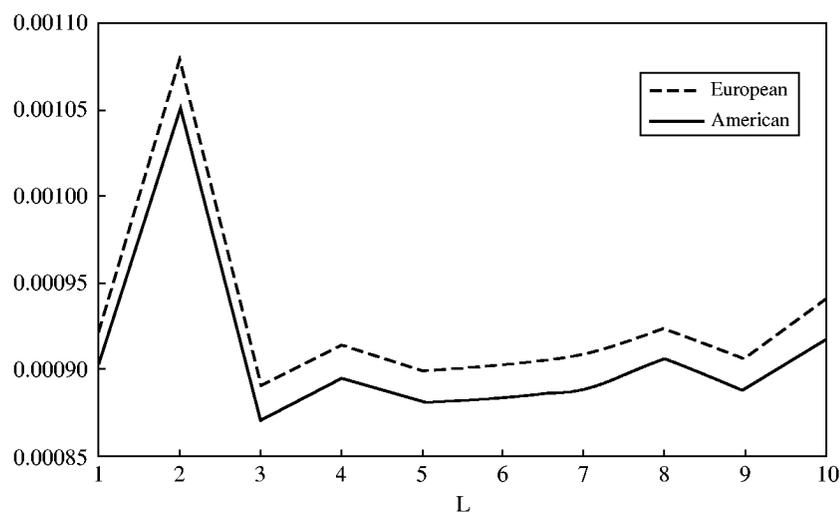


FIGURE 5

Price difference of GMAL and AMAL. Both European-style and American-style plots are considered. The parameters are $S_0 = UB = 50$, $\sigma = 40\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), and $a = 3$.

SOME GREEKS

Delta and gamma are key to risk management and hedging. Because of the price similarity between GMAL and AMAL, our main focus is on EGMALS. The general observations made below apply to all MALs, however. The sensitivity of MAR is also studied. The formulae to compute the sensitive measures are from Pelsser and Vorst (1994).

Delta

Figure 6 illustrates option prices with different combinations of LB and UB. In the extreme case when $UB = LB$, EGMAL reduces to a vanilla call. The price difference between the vanilla call and EGMAL diminishes as S_0 increases. When S_0 decreases, however, the reset feature makes the EGMAL's price decrease more slowly than the vanilla call's. An interesting feature of the figure is the concavity of the EGMAL price when the stock price falls between LB and UB. It suggests nonmonotonicity in the value of delta, unlike the vanilla call. This phenomenon reflects the two forces determining the option value. As the stock price decreases, the downward stock price tends to drag down the option value, whereas the prospect of a lower strike price tends to raise the option value. When the stock price is between LB and UB, the strike-price effect is at its strongest and is able to counteract the decrease in the option value. The option holder is therefore protected by the downward-reset feature. When the stock price

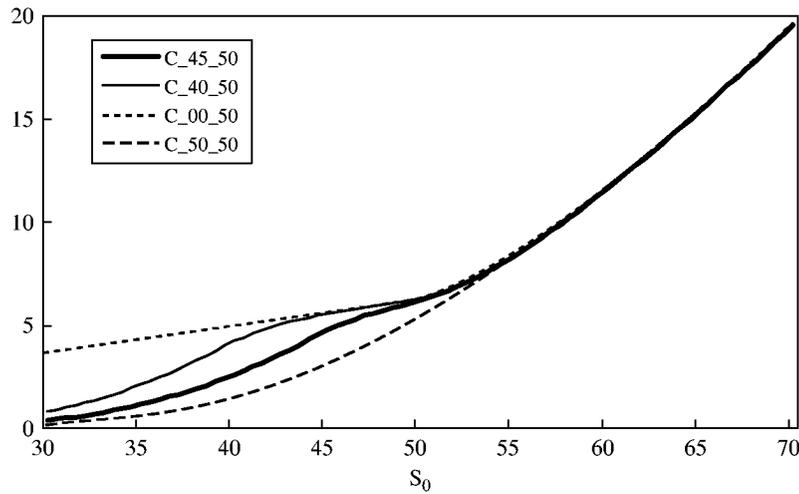


FIGURE 6
 Dependency of EGMA's price on lower and upper bounds. The parameters are $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $L = 3$, and $a = 3$. C_{x_y} means EGMA comes with $LB = x$ and $UB = y$.

penetrates below LB, however, the stock-price effect dominates, and the protection ceases to exist.

Figure 7 explores this issue with the delta information. Indeed, the delta of EGMA is not a monotonically increasing function of S_0 . Looked at more closely, the delta is a decreasing function of the stock price when the stock price is roughly between LB and UB.

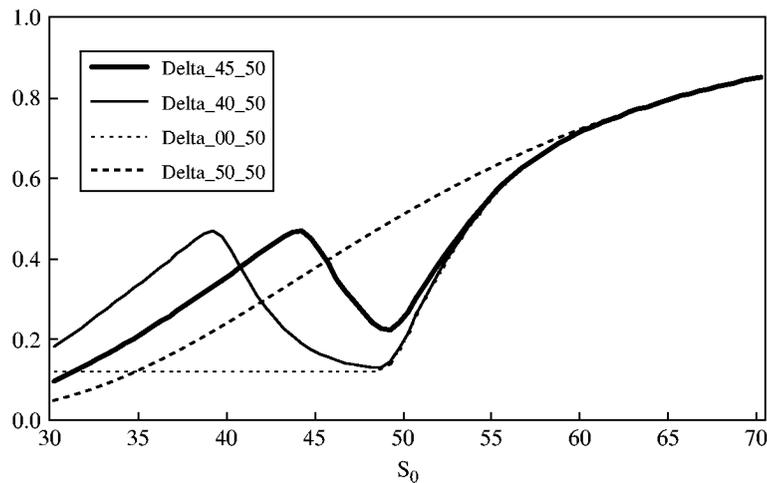


FIGURE 7
 Delta of EGMA. The parameters are $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $L = 3$, and $a = 3$. Δ_{x_y} means $LB = x$ and $UB = y$.

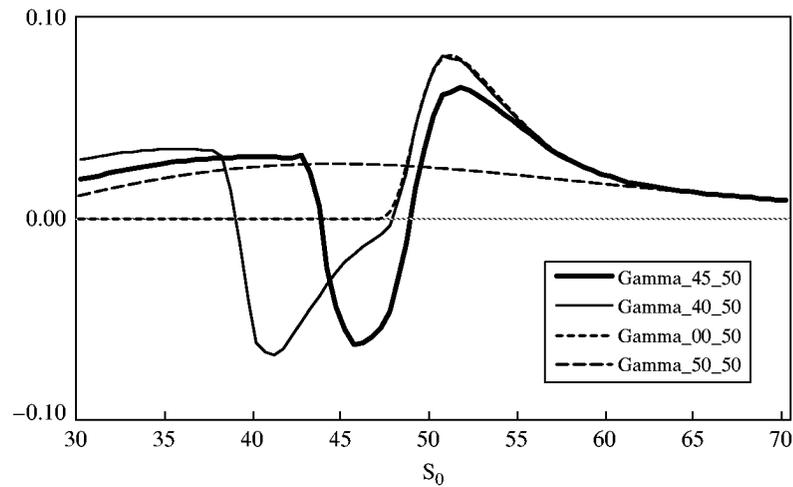


FIGURE 8

Gamma of EGMAL. The parameters are $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $L = 1$, and $a = 3$. Gamma_ x _ y means LB = x and UB = y .

Gamma

Delta's complex behavior suggests a similar case for gamma, and Figure 8 confirms that. Gamma is negative roughly between LB and UB; for vanilla European-style calls, in contrast, gamma is always positive. When a securities firm writes a derivative security with a positive gamma, the firm's position has a negative gamma. In that case, the delta hedge strategy will result in hedging losses—the so-called gamma risk. A delta–gamma hedge strategy may be followed instead. This is especially relevant when the underlying stock price is volatile or when the hedging frequency is low. The opposite is true when the derivative has a negative gamma. Apparently, issuing a derivative with negative gamma benefits the issuer.

MAL without an LB is an extreme case. Gamma is essentially zero for $S_0 < UB$, which implies that the option value decreases with the underlying stock price in roughly a linear manner. This reflects the fact that the strike price can be continuously adjusted down to zero.

Delta Jumps of MAR

One more parameter N_s is added in the specification of MARS. MAR has N_s reset strike prices set linearly between LB and UB

$$UB - h, UB - 2h, \dots, LB$$

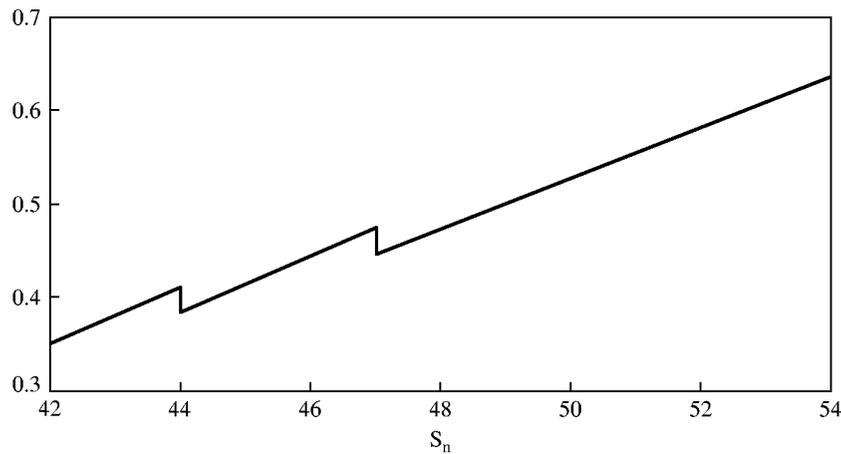


FIGURE 9

Delta of AMAR at reset date. The parameters are $S_{n-2} = S_{n-1} = 50$, $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 11/12$, $n = 0$, $a = 3$, $LB = 48$, $UB = 50$, and $N_s = 2$ (thus, reset strike prices are 49 and 48). The prevailing strike price is 50.

where $h \equiv (UB - LB)/N_s$. The strike price is changed to the lowest reset strike price that the prevailing moving average touches or penetrates.

Like most reset options, the feature of discrete reset strike prices makes MAR witness a delta jump at the stock price such that the strike price is lowered. Therefore, MAR can be very sensitive to the stock price. Consider an arithmetic MAR (AMAR) at reset date n . The prevailing strike price is 50, $a = 3$, $S_{n-2} = S_{n-1} = 50$, $LB = 48$, $UB = 50$, and $N_s = 2$. The reset strike prices are thus 49 and 48. Clearly, the strike price will be revised down to 49 if $44 < S_n \leq 47$, and down to 48 if $S_n \leq 44$. This results in two kinks for the option value and two delta jumps at $S_n = 47$ and $S_n = 44$ as shown in Figure 9.

Because the delta jump depends on the past $a - 1$ stock prices, it makes the stock price at the reset date, S_n , play a less critical role in MARS than for ordinary reset options, whose strike price depends solely on the stock price. The probability of a delta jump at the reset date is also smaller. This is yet another advantage of MARS.

EMPIRICAL STUDIES

Now that the performance of the algorithms has been ensured and sensitivity measures examined, this article uses them for empirical studies in pricing and delta hedge effectiveness. Two American-style AMALS (PL06 and PL07 issued by Polaris Securities) and two American-style arithmetic

TABLE VII
Contract Specifications of PL06, PL07, GC06, and NS02

#	PL06	PL07	GC06	NS02
Type	Lookback	Lookback	Reset	Reset
Issue date	08/21/99	08/27/99	04/28/99	06/16/99
First trading date	09/02/99	09/06/99	05/12/99	06/30/99
Reset date	09/20/99	09/27/99	08/11/99	07/15/99
Expiration date	09/01/00	09/05/00	05/11/00	06/29/00
Moving-average period (days)	6	6	6	3
UB	103.75	64.45	81.00	81.30
LB	$0.9 \times \text{UB}$	$0.9 \times \text{UB}$	$0.9 \times \text{UB}$	$0.9 \times \text{UB}$
Number of reset strike prices	—	—	5	5
Issue volatility	54.38%	54.58%	49.10%	50.43%
Issue price (TWD)	26.93	16.76	20.25	20.00

Note. Information is provided by the issuing firms.

MARS (GC06 and NS02 issued by Grand Cathay Securities and National Securities, respectively) are the main targets.

The contracts are specified in Table VII, and the ensuing parameter setups for the algorithms are tabulated in Table VIII. The prices calculated by the algorithms, in Table IX, are essentially identical to the issue prices: $26.81 \approx 26.98$, $16.67 \approx 16.76$, $19.88 \approx 20$, and $19.89 \approx 20.25$. The relative differences between the implied volatilities calculated by the algorithms and those released by the securities firms are less than 1%. The GMAL approximations are also very tight for PL06 and PL07. Prices calculated by both AMAL and GMAL algorithms are within one standard error of the Monte Carlo simulation on the basis of

TABLE VIII
Parameter Setups for PL06, PL07, GC06, and NS02 for the Tree Algorithms

#	PL06	PL07	GC06	NS02
S_0	103.75	64.45	81.00	81.30
UB	103.75	64.45	81.00	81.30
LB	93.38	58.01	72.90	73.17
σ	54.38%	54.58%	49.10%	50.43%
r	5.00%	5.00%	5.00%	5.00%
q	0.00%	0.00%	0.00%	0.00%
T	378/365	376/365	380/365	380/365
T_s	31/365	32/365	105/365	30/365
n	24	24	81	21
a	6	6	6	3
L	2	2	2	11
N_s	—	—	5	5

TABLE IX
Prices and Volatilities of PL06, PL07, GC06, and NS02

#	AMAL		AMAR	
	PL06	PL07	GC06	NS02
CRR	26.8125	16.6689	19.8866	19.8841
GMAL approximation (CRR)	26.8181	16.6725	—	—
MC	26.8160	16.6714	19.9003	19.8786
Standard error	0.0071	0.0045	0.0104	0.0050
Implied volatility by CRR	54.80%	54.95%	49.50%	50.78%

Note. The parameters are based on Table VIII. Recall the preceding proof that says MALS and MARS will not be exercised early when options are dividend-protected, as is the case here. Monte Carlo simulation thus gives valid results even if MAL and MAR above are American style.

2,000,000 paths (1,000,000 plus 1,000,000 antithetic). The tightness of the results gives us confidence in their respective correctness.

The almost exact match of the issue prices and those computed by our algorithms is not hard to explain. After all, it has been previously confirmed that the options under study will not be exercised early. Hence, they cost the same as their European-style counterparts. Treating them as European-style options and pricing them accordingly with Monte Carlo simulation therefore give the same prices. This claim does not hold, however, if the options are not dividend-protected.

The implied volatilities of PL06 and PL07 in Figures 10 and 11 are calculated with the AMAL algorithm up to September 18, 1999.³ The two plots suggest that the volatilities lie within the range between 55 and 65%. This level of volatility is typical of stocks in Taiwan. Interestingly, there is obvious divergence of movements between implied volatilities and the underlying stock prices in both figures. Because these options are issued in fixed quantities, they may carry a fixed-supply premium. This feature also makes short selling somewhat constrained.

Delta hedge for a short option position is investigated for the two American-style AMALS: PL06 and PL07. Daily hedging is adopted. The experiments show that, on September 20, 1999, a hedge loss of 2.24 TWD (−8.30%) is incurred on PL06, and a hedge loss of 0.06 TWD (−0.03%) is incurred on PL07. This is probably due to the increase in volatilities during this period. The issue volatilities are 54.80 and 54.95% for PL06 and PL07, respectively, whereas the implied volatilities on

³Because of a major earthquake on September 21, 1999, the Taipei Stock Exchange was shut down from September 21 to September 26. Therefore, the implied volatilities in Figure 11 stop on September 20, 1999. After September 27, all the options were no longer resettable (review Table VII), having become vanilla calls.

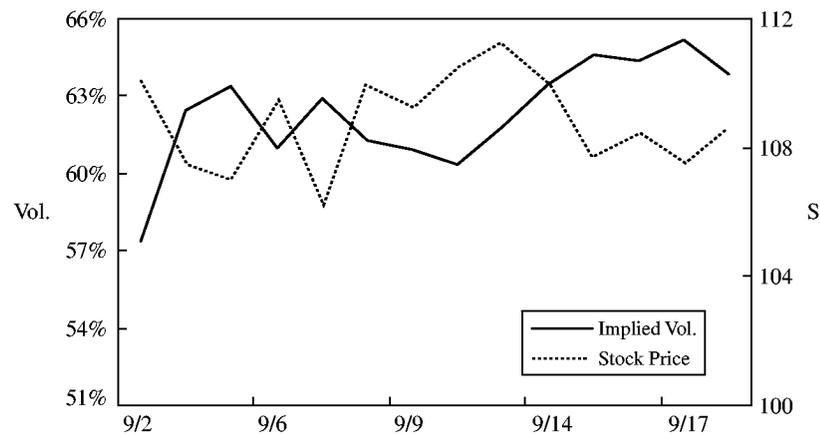


FIGURE 10
Implied Volatilities of PL06.

September 20, 1999, are 63.82 and 58.24%. Also, during the period the prices of the underlying asset of PL06 exceed UB, whereas those of PL07 move between LB and UB. The previous section implies that these price behaviors will disadvantage the hedging of PL06 but benefit the hedging of PL07 because of negative gamma. These inferences are consistent with the aforementioned findings.

CONCLUSIONS

This article develops a general methodology for pricing moving-average-type options. In particular, it gives practical algorithms for the moving-average-lookback option and the moving-average-reset option. Both

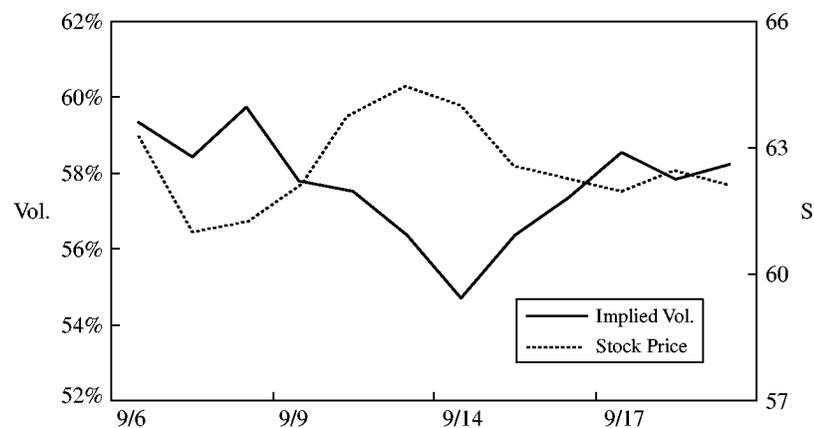


FIGURE 11
Implied Volatilities of PL07.

options were traded on Taiwan's exchange, for example. The algorithms converge quickly to the correct value as verified by simulation. Interestingly, the price difference between the geometric and arithmetic moving-average-lookback options is very small. Because it takes much less time to price the geometric version, it is a good approximation to the arithmetic version. The delta and gamma of these options were investigated and possessed complex behavior. This phenomenon has implications for hedging. The algorithms were tested empirically against those moving-average-type options traded on Taiwan's stock exchange. The results show surprisingly tight matches in the issue prices and implied volatilities supplied by the issuers. When applied to delta hedge, the algorithms indicate conclusions consistent with those suggested by the numerical results.

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